QED COUPLING FROM A 5-D KALUZA-KLEIN THEORY

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We discuss the possibility to obtain, from a five-dimensional free spinor Lagrangian, the Quantum Electro-Dynamics (QED) coupling via a Kaluza-Klein reduction of the theory. This result is achieved taking a phase dependence of the spinor field on the extra-coordinate and modifying the corresponding connection. The five-dimensional spinor theory is covariant under the admissible coordinates transformations and its four-dimensional reduction provides the QED coupling term.

We assume, as a starting point, a five-dimensional smooth manifold $V^4 \otimes S^1$ endowed with a metric $\mathbf{j}$ of components j_{AB}.
We make the hypothesis that for all physical quantities there is no dependence on the fifth coordinate x^5; in particular for the metric we have that $j_{AB} = j_{AB}(x^\mu)$, $(\mu = 0, 1, 2, 3)$.
Within a Kaluza-Klein theory the admissible coordinates transformations reduce to the form $x^{\mu'} = x^{\mu'}(x^\nu) \quad x^{5'} = x^5 + \alpha(x^\nu)$, according to which the component j_{55} behaves like a scalar field and below it will be taken equal to unity. Then, the components $j_{5\mu}$ can be identified with a vector field (the electromagnetic potential A_μ) and the five-dimensional Einstein-Hilbert action splits into the four-dimensional Einstein-Maxwell one. In this theory, the components $j_{\mu\nu}$ differ from the four-dimensional metric tensor $g_{\mu\nu}$ by a term $\propto A_\mu A_\nu$.

We investigate how to get, via a splitting procedure, the electrodynamics coupling term starting from a free spinor living on a Kaluza-Klein space-time. Let us introduce in the five-dimensional space-time a spinor field as a matter one, for which we take the following Lagrangian density (expressed in "tetradic" form)

$$ {}^5\Lambda = -\frac{i\hbar c}{2}\bar{\chi}\gamma^{(A)}D_{(A)}\chi + \frac{i\hbar c}{2}(D_{(A)}\bar{\chi})\gamma^{(A)}\chi + imc^2\bar{\chi}\chi \,, \tag{1} $$

where D is the new covariant derivative, endowed with the spinor connection Γ_μ

$$ \Gamma_\mu = -\frac{1}{4}\gamma^\nu \nabla_\mu \gamma_\nu \tag{2} $$

to preserve the parallel transport of Dirac's matrices.
Now we split the action associated to the four-dimensional quantities and achieve the dimensional reduction which provides the QED one. Firstly we express the spinor field χ and the 'tetradic' spinor connection in terms of four dimensional quantities. According to the cylindricity condition, the spinor field is chosen as independent of the fifth coordinate, assuming the standard form (2) for the spinor connection i.e.

$$\begin{cases} \chi = \frac{1}{\sqrt{L}}\psi \\ \\ \bar{\chi} = \frac{1}{\sqrt{L}}\bar{\psi} \end{cases} \tag{3}$$

where ψ is the four-dimensional Dirac's field, L the extra-dimension length and

$$\begin{cases} \Gamma_{(\mu)} = {}^4\Gamma_{(\mu)} - \frac{ek}{4}\gamma^{(5)}\gamma^{(\rho)}F_{(\mu)(\rho)} \\ \\ \Gamma_{(5)} = -\frac{ek}{8}\gamma^{(\alpha)}\gamma^{(\rho)}F_{(\alpha)(\rho)} \end{cases} \tag{4}$$

being $F_{(\alpha)(\rho)}$ the electromagnetic tensor, $\gamma^{(A)}$ the Dirac's matrices of flat space and all the quantities have been expressed in the 'tetradic' form.
Under these assumptions, we do not obtain the correct QED coupling; in fact the splitted action reads as

$$S = \int \sqrt{-g}\left[-\frac{i\hbar}{2}\bar{\psi}\gamma^{\mu}D_{\mu}\psi + \frac{i\hbar}{2}(D_{\mu}\bar{\psi})\gamma^{\mu}\psi + imc\bar{\psi}\psi + \frac{i\hbar ek}{8}\bar{\psi}\gamma^{(5)}\gamma^{\mu}\gamma^{\nu}\psi F_{\mu\nu}\right] d^4x\,, \tag{5}$$

where e is the electric charge and k is a dimensional constant. In (5) there is no coupling between the electromagnetic potential and the spinor current J_{μ}, while it appears a new term that has no clear physical interpretation; since such a term survives also on a flat space-time, then the present approach is wrong because it predicts non-observed electrodynamics couplings.

Thus now we pursue an alternative point of view, observing that the spinor is a wave function and therefore admits a phase dependence on the fifth coordinate, such as

$$\chi \equiv \frac{1}{\sqrt{L}}e^{i\frac{2\pi x^5}{L}}\psi(x^{\nu}) \qquad \bar{\chi} \equiv \frac{1}{\sqrt{L}}e^{-i\frac{2\pi x^5}{L}}\psi(x^{\nu}). \tag{6}$$

This choice for the spinor field is a good one, as confirmed by considering that we obtain an equivalence between the charge operator and the fifth component of the five-momentum operator; it can be shown using the Noether's theorem.
It is just the phase term in the spinor field which allows the appearance of the expected QED coupling.
In spite of this success, we still have the problem to avoid a misleading term; in fact the phase introduces a contribution containing the matrix $\gamma^{(5)}$ in the splitted

action. This is eliminated by choosing a different form for the spinor connection as

$$\begin{cases} \Gamma_{(\mu)} =^4 \Gamma_{(\mu)} \equiv -\frac{1}{4}\gamma^{(\rho)}\gamma^{(\sigma)}R_{(\sigma)(\rho)(\mu)} & \text{greek indexes go from 1 to 4} \\ \Gamma_{(5)} = \frac{2\pi i}{L}\mathbf{I} & \text{with } \mathbf{I} \text{ identity matrix.} \end{cases} \quad (7)$$

It is worth stressing that this form for the spinor connection leaves the Lagrangian density (1) invariant under the restricted transformation of coordinates allowed in a Kaluza-Klein theory.

In terms of the new connection, the relations $D_A\gamma_B = 0$ no longer hold, but we preserve the validity of the four-dimensional conditions ${}^4D_\mu{}^4\gamma_\nu = 0$.

By other words the Dirac's algebra is still valid in the four-dimensional space-time once we have carried out the dimensional reduction on our action by integrating over the fifth dimension.

Working out the (1) we obtain

$${}^5\Lambda = -\frac{i\hbar c}{2}\bar{\chi}\gamma^\mu{}^4D_\mu\chi - \frac{i\hbar c}{2}({}^4D_\mu\bar{\chi})\gamma^\mu\chi - \frac{2\pi ek\hbar c}{L}A_\mu\bar{\chi}\gamma^\mu\chi + imc^2\bar{\chi}\chi\,. \quad (8)$$

Starting from the five-dimensional action, integrating on x^5 and using the cylindricity condition as well as the expression of χ and $\bar{\chi}$, we get the total splitted action

$$S = \frac{1}{c}\int\sqrt{-g}\Big[-\frac{c^4}{16\pi G}\widehat{R} - \frac{1}{16\pi}F_{\mu\nu}F^{\mu\nu} - \frac{i\hbar c}{2}\bar{\psi}\gamma^\mu D_\mu\psi + \frac{i\hbar c}{2}(D_\mu\bar{\psi})\gamma^\mu\psi - eA_\mu\bar{\psi}\gamma^\mu\psi + imc^2\bar{\psi}\psi\Big]\,d^4x\,, \quad (9)$$

where we included also the part purely geometric of the action and taking

$$\begin{cases} k = \frac{\sqrt{4G}}{ec^2} \\ L = 2\pi\sqrt{4G}\,\frac{\hbar}{ec} = 4.75\,10^{-31}\text{cm}\,; \end{cases} \quad (10)$$

to get the terms $\frac{1}{16\pi}F_{\mu\nu}F^{\mu\nu}$ and $eA_\mu\bar{\psi}\gamma^\mu\psi$ in the action. The estimation of L is in agreement with other well-known ones present in literature.

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